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On the supersymmetric effective action of Matrix theory

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Abstract

We present a simple derivation of the supersymmetric one-loop effective action of SU(2) Matrix theory by expressing it in a compact exponential form whose invariance under supersymmetry transformations is obvious. This result clarifies the one-loop exactness of the leading $(v^2)^2$ interactions and the absence of non-perturbative corrections.

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Recently maximally supersymmetric $SU(N)$ gauge quantum mechanics in $d = 9$ [1] has gained prominence due to its relation to the low-energy dynamics of zerobranes in type IIA string theory [2], the close relation between its $N \to \infty$ limit and the eleven-dimensional supermembrane [3], as well as the M theory proposal of [4]. A key feature of this model is the existence of flat directions in the Cartan sector on which scattering states localize. To date almost all investigations of scattering amplitudes in Matrix theory make use of the perturbative construction of an effective Lagrangian for the Cartan valley degrees of freedom at finite N, which is based on a loopwise expansion around the solution of the classical equations of motions, $\ddot{x}^i_I = 0$ and $\dot{\theta}_I = 0$, where $I, J, \ldots = 1, ..., N-1$. Although this approach simply ignores contributions from bound states *all* tree level amplitudes computed to date in matrix theory agree with the results obtained from eleven dimensional supergravity [5, 6]. As soon as one goes beyond the tree level regime, however, this correspondence breaks down[7][1]. As argued in [8, 9] the agreement of the $(v^2)^2$ and $(v^2)^3$ terms in the effective action with tree level supergravity could be solely due to the high amount of supersymmetry in the problem. In particular in [8] it was shown that the leading corrections to the $SU(2)$ effective action of order $(v^2)^2$ are completely determined by supersymmetry, a claim thereafter made explicit by [10]. In this note we wish to demonstrate how this rather involved analysis may be condensed to a two line argument, yielding the complete form of the supersymmetric $SU(2)$ effective action at order $(v^2)^2$.

The basic variables for the $SU(2)$ theory in the Cartan subalgebra are

$$X^i(t) \quad , \quad v^i(t) := \dot{X}^i(t) \qquad \text{and} \qquad \theta_\alpha(t) \tag{1}$$

where $i, j, ... = 1, ..., 9$ and $\alpha, \beta = 1, ..., 16$. These variables correspond to the diagonal degrees of freedom in the matrix theory; for $SU(N)$ we would have X^i_I with $I = 1, \ldots, (N-1)$.

The supersymmetry variations are given by:

$$\delta X^i = -i\epsilon\gamma^i\theta + \epsilon N^i\theta \qquad \delta\theta = v^j\gamma_j\epsilon + M\epsilon \tag{2}$$

where N^i and M correspond to higher order modifications. For $N^i = M = 0$, these variations leave invariant the free action

$$S^{(0)} = \int dt \left[\frac{1}{2}v^2 + \frac{i}{2}\theta\dot{\theta}\right] \tag{3}$$

Besides these terms, the effective action will contain an infinite string of higher order corrections. Since the algebra closes only on-shell, the supersymmetry variations must be modified accordingly such that N^i and M will no longer vanish.

[1] In fact, it can be shown that there is *no* Lorentz invariant combination of R^4 terms that reproduces the matrix theory result.

In considering such corrections, one must also take into account nonlinear field redefinitions

$$X^i \longrightarrow X'^i = X'^i(X,v,\theta) \qquad \theta_\alpha \longrightarrow \theta'_\alpha = \theta'_\alpha(X,v,\theta) \tag{4}$$

Modifications of the supersymmetry variations induced by such redefinitions do preserve the algebra, but should be discarded as they do not correspond to genuine deformations of the original variations.

Remarkably, even in this simple quantum mechanical context, no nontrivial modifications with $N^i, M \neq 0$ have so far been explicitly exhibited in the literature, although in [9] evidence for non-trivial $N^i \sim \theta^4$ and $M \sim \theta^6$ modifications was presented. A full treatment is difficult because a complete analysis of the superalgebra and its closure will presumably require the consideration of infinitely many corrections. The problem is aggravated by the fact that for the maximally extended models no off-shell formulation is known[2].

To simplify matters, one makes the assumption that

$$\frac{dv^i}{dt} = 0 \quad , \quad \frac{d\theta}{dt} = 0 \tag{5}$$

This assumption, which implicitly also underlies all previous work, allows us to drop all derivatives other than those of X^i in the variations, and greatly simplifies the calculation; for instance, we can then consistently set $\delta v^i = 0$ for the unmodified variations. Effectively, the above condition amounts to a reduction of a quantum mechanical system to a "zero-dimensional" system. The freedom of making field redefinitions is reduced accordingly: the only redefinitions compatible with the above reduction are the ones preserving the linear dependence of X^i on t and the constancy of v^i and θ.

In [11, 6, 12, 10] the full supersymmetric one-loop effective action was shown to be

$$\begin{aligned} S^{(1)} \;=\; & \int dt \Big[(v^2)^2 f(X) + \frac{i}{2} v^2 v^j \partial_j f \,(\theta\gamma^{ij}\theta) - \frac{1}{8} v^i v^j \partial_k \partial_l f(X)\,(\theta\gamma^{ik}\theta)(\theta\gamma^{jl}\theta) \\ & - \frac{i}{144} v^i \partial_j \partial_k \partial_l f(X)\,(\theta\gamma^{ij}\theta)(\theta\gamma^{km}\theta)(\theta\gamma^{lm}\theta) \\ & + \frac{1}{8064} \partial_i \partial_j \partial_k \partial_l f(X)\,(\theta\gamma^{im}\theta)(\theta\gamma^{jm}\theta)(\theta\gamma^{kn}\theta)(\theta\gamma^{ln}\theta) \Big] \end{aligned} \tag{6}$$

Here $f = f(X)$ is a harmonic function, i.e. $\triangle f \equiv \partial_j \partial_j f(X) = 0$ with the unique rotationally invariant solution $f = r^{-7}$ (where $r := \sqrt{X^i X^i}$). Provided one assumes constancy of v^i and θ the action $S^{(1)}$ must be invariant under the

[2] Besides, it is doubtful whether an off-shell formalism would be of much use here, as the "rules of the game" are no longer clear: the elimination of auxiliary fields via their equations of motion and via the path integral yield inequivalent results unless the auxiliary fields appear at most quadratically in the Lagrangian.

unmodified supersymmetry variations above, as the modified variation of the free action $S^{(0)}$ under constant v^i and θ

$$\delta S^{(0)} = \int dt\, \partial_t \left(v^i\, \epsilon N^i \theta \right) \tag{7}$$

vanishes for asymptotically suppressed corrections, $\lim_{t\to\pm\infty} N^i = 0$. Possible leading modifications of the supersymmetry transformations were discussed in [10], but clearly these do not contribute under the assumption (5).

We will now show that the complicated action $S^{(1)}$ can be cast into a much simpler form, whose supersymmetry invariance is very easy to check. Namely, we have

$$\begin{aligned} S^{(1)} &= \int dt\, (v^2)^2 \exp\left[\frac{i}{2v^2}\theta\gamma^{ij}\theta v_i \partial_j\right] f(X) \\ &= \int dt\, (v^2)^2 f\left(X - \frac{i}{2v^2}\theta\gamma^{ij}\theta v_j\right) \end{aligned} \tag{8}$$

To prove that this action indeed coincides with (6), we first observe that in the above action we can neglect all terms containing the Laplacian (which annihilates f) as well as terms containing $v^j\partial_j$, because with constant v^i and θ, this term can be pulled out, yielding a total time derivative. To proceed, we show that

$$(\theta\gamma^{ij}\theta v_i\partial_j)(\theta\gamma^{kl}\theta\partial_l)(\theta\gamma^{km}\theta\partial_m) \simeq \frac{3}{v^2}(\theta\gamma^{ij}\theta v_i\partial_j)^3 \tag{9}$$

and

$$\Big((\theta\gamma^{kl}\theta\partial_l)\, (\theta\gamma^{km}\theta\partial_m) \Big)^2 \simeq \frac{21}{(v^2)^2}\, (\theta\gamma^{ij}\theta v_i\partial_j)^4 \tag{10}$$

from which the equivalence follows up to fourth order. The symbol $\simeq$ here and below means equality modulo contributions containing $v^i\partial_i$ or $\triangle$. To verify the above relations we start out from the Fierz identity (see e.g. [10] for a comprehensive list of Fierz identites)

$$(\theta\gamma^{ijk}\theta\, v_i\partial_j)^2 \simeq -5\, (\theta\gamma^{ij}\theta\, v_i\partial_j)^2 + v^2\, (\theta\gamma^{ij}\theta\partial_j)\, (\theta\gamma^{ik}\theta\partial_k) \tag{11}$$

Thereafter one multiplies (11) with $(\theta\gamma^{ij}\theta\, v_i\partial_j)$ so that its left hand side may be rewritten as

$$(\theta\gamma^{ij}\theta\, v_i\partial_j)\, (\theta\gamma^{klm}\theta\, v_k\partial_l)^2 \simeq -\tfrac{1}{6}v^2\, (\theta\gamma^{ij}\theta\partial_i)\, (\theta\gamma^{klj}\theta\, \partial_k)\, (\theta\gamma^{mnl}\theta\, v_m\partial_n)\,. \tag{12}$$

upon using yet another Fierz identity. Now once more perform a Fierz rearrangement on the last two terms of the above expression to obtain

$$(\theta\gamma^{ij}\theta\, v_i\partial_j)\, (\theta\gamma^{klm}\theta\, v_k\partial_l)^2 \simeq -\tfrac{2}{3}v^2\, (\theta\gamma^{ij}\theta\, v_i\partial_j)\, (\theta\gamma^{kl}\theta\, \partial_k)^2 \tag{13}$$

This is to be contrasted with the right hand side of (11) multiplied with $(\theta\gamma^{ij}\theta\, v_i\partial_j)$:

$$(\theta\gamma^{ij}\theta\, v_i\partial_j)\, (\theta\gamma^{klm}\theta\, v_k\partial_l)^2 \simeq -5\, (\theta\gamma^{ij}\theta\, v_i\partial_j)^3 + v^2\, (\theta\gamma^{ij}\theta\, v_i\partial_j)\, (\theta\gamma^{kl}\theta\partial_k)^2 \tag{14}$$

From (13) and (14) the relation (9) immediately follows. Relation (10) is then shown in a similar manner.

Next we note that the exponential series of (8) terminates already after the fourth order term because

$$(\theta\gamma^{ij}\theta v_i\partial_j)^5 \simeq 0 \quad , \tag{15}$$

which is an immediate consequence of (9) and (10):

$$\begin{aligned}
\frac{21}{(v^2)^2}\,(\theta\gamma^{ij}\theta v_i\partial_j)^5 &\simeq (\theta\gamma^{ij}\theta v_i\partial_j)\left((\theta\gamma^{kl}\theta\partial_l)\,(\theta\gamma^{km}\theta\partial_m)\right)^2 \\
&\simeq \frac{3}{v^2}(\theta\gamma^{ij}\theta v_i\partial_j)^3\,(\theta\gamma^{kl}\theta\partial_l)\,(\theta\gamma^{km}\theta\partial_m) \\
&\simeq \frac{9}{(v^2)^2}(\theta\gamma^{ij}\theta v_i\partial_j)^5
\end{aligned} \tag{16}$$

where we used (10) in the first and (9) in the second and third lines. Hence there is no need to truncate the series (it would anyhow terminate at order θ^{16} by the nilpotency of the Grassmann variables).

The supersymmetry of this action with the above assumptions (5) (and $N^i = M = 0$) can now be proven in two lines:

$$\begin{aligned}
\delta S^{(1)} &= \int dt(v^2)^2 \exp\left[\frac{i}{2v^2}\theta\gamma^{ij}\theta v_i\partial_j\right]\left(\frac{i}{v^2}\delta\theta\gamma^{ij}\theta v_i\partial_j f(X) + \delta X^i\,\partial_i f(X)\right) \\
&= -\int dt(v^2)^2 \exp\left[\frac{i}{2v^2}\theta\gamma^{ij}\theta v_i\partial_j\right]\,\frac{i}{v^2}\,\epsilon\,\not{v}\,\theta\, v^i\,\partial_i f(X) = 0\,.
\end{aligned} \tag{17}$$

Remarkably, this simple argument works *for any action* of the form

$$\int dt \exp\left[\frac{i}{2v^2}\theta\gamma^{ij}\theta v_i\partial_j\right] g(X,v) \tag{18}$$

and in particular yields supersymmetric completions of

$$g(r,v) = \frac{(v^2)^m}{r^n} \tag{19}$$

Uniqueness and therefore a "non-renormalization theorem" holds only for actions with low powers of v^2, and only if one insists on the absence of terms singular in v^2. For $g(r,v) \propto v^2$, the only way to avoid such singular terms is to require $\partial_i g = 0$, in which case one is left with the free action only. For $g(r,v) \propto (v^2)^2$, a singularity could arise at order θ^6, and is eliminated by means of the requirement $\triangle g = 0$ (and the above Fierz identities implying (15)). Unfortunately for $g(r,v) \propto (v^2)^3$ our above arguments fail, as the *modified* supersymmetry transformations of $S^{(1)}$ now do produce non-vanishing terms under the constancy assumption of v^i and θ. Hence even in this reduced sector (18) cannot be the full answer. At order

$g(r,v) \propto (v^2)^4$ and beyond, no singular terms can arise, and the choice of $g(r,v)$ is not restricted in any way. This is meant by the statement that, at this order there is no "non-renormalization theorem".

We have thus seen that the one-loop effective action (6) resp. (8) laboriously computed in [11, 6, 12, 10] is indeed completely fixed by supersymmetry. Therefore the agreement of the resulting spin dependent Matrix theory scattering amplitudes with tree level supergravity [6, 12] does not test any dynamical aspects of the M theory proposal, but is solely due to the right amount of supersymmetry in the problem.

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